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WEAK CONVERGENCE OF THE SIMPLE LINEAR RANK STATISTIC UNDER MIXING CONDITIONS IN THE NONSTATIONARY CASE*

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Abstract. The asymptotic distribution theory of simple linear rank statistics for the case when the underlying random variables are nonstationary is studied both for the φ-mixing and strong mixing cases.

Key words. weak convergence; linear rank statistics, empirical processes, φ-mixing, strong mixing

1. Introduction. Let X_{ni}, $1 \leqq i \leqq n$, $n \geqq 1$, be real-valued r.v.'s (random variables) with continuous d.f.'s (distribution functions) $F_{ni}(x)$, $x \in \mathbf{R}$, and let c_{ni} $(1 \leqq i \leqq n,\ n \geqq 1)$ be an array of regression constants defined by a function g on $[0, 1]$ as

$$c_{ni} = g(i/n), \qquad 1 \leqq i \leqq n, \quad n \geqq 1. \tag{1.1}$$

Denote by $\widehat{H}_n(x) = n^{-1} \sum_{i=1}^n c_{ni} \mathbf{1}_{[X_{ni} \leqq x]}$ the weighted empirical process where $\mathbf{1}_{[\,]}$ denotes the indicator function and by $\widehat{F}_n(x) = n^{-1} \sum_{i=1}^n \mathbf{1}_{[X_{ni} \leqq x]}$ the usual empirical process.

The corresponding expectations are denoted by

$$H_n = \mathbf{E}\big(\widehat{H}_n\big) \quad \text{and} \quad F_n = \mathbf{E}\big(\widehat{F}_n\big).$$

We will study the asymptotic behavior of the simple linear rank statistic of the form

$$\mathcal{S}_n(J) = n^{1/2} \int_{-\infty}^{+\infty} J\Big(\frac{n}{n+1}\, \widehat{F}_n(x)\Big)\, d\widehat{H}_n(x) \int_{-\infty}^{+\infty} J\big(F_n(x)\big)\, dH_n(x), \tag{1.2}$$

where J is a score function defined on the open unit interval.

The problem of finding a sufficiently large class of score functions for which the linear rank statistic is asymptotically normal was first considered by Chernoff and Savage [3]. Their results were later on strengthened considerably by several authors, mainly by Govindarajulu et al. [8], Pyke and Shorack [10], Hájek [9] and Dupač and Hájek [6] for the independent case, by Fears and Mehra [7] for the φ-mixing case with stationary random variables, and by Denker and Rösler [4] for the φ-mixing as well as strong mixing case but under a *stationary* set-up. In this paper we investigate the asymptotic distribution theory of the simple linear rank statistics (1.2) for the case when the underlying random variables are *nonstationary*.

2. Preliminaries. In this section, we give some propositions that are minor variations of Denker and Rösler [4] and so their proofs will be either omitted or briefly outlined.

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For $\delta \geqq 0$, set $\eta = \delta(4+2\delta)^{-1}$.

Let μ_δ denote the measure on [0, 1] given by its density $(z(1-z))^{-1/2-\eta}$ relative to the Lebesgue measure. For a monotone function J, let $\|J\|_\delta$ be the $\mathbf{L}_1$-norm of J in $\mathbf{L}_1(\mu_\delta)$. By the Jordan decomposition of any right continuous function, J has a unique decomposition: $J = J_1 - J_2$ where J_1 and J_2 are monotone functions and $J_1(\frac{1}{2}) = 0$. For such a function, we set

$$\|J\|_\delta = \|J_1\|_\delta + \|J_2\|_\delta,$$

where J_1 and J_2 belong to $\mathbf{L}_1(\mu_\delta)$.

Denote by $\mathcal{H}_\delta$ the space of all right continuous functions J with $\|J\|_\delta < \infty$ and $J(\frac{1}{2}) = 0$ and let $\mathcal{G}_\delta$ be the set of all $J \in \mathcal{H}_\delta$ for which the measure ν defined by $J = \int d\nu$ is absolutely continuous with respect to Lebesgue measure. It is well known that $\mathcal{G}_\delta$ is the $\|\ \|_\delta$-norm closure of $\mathbf{C}_{2,b}$: the space of functions with bounded second derivatives. The a priori assumption of having the space $\mathcal{H}_\delta$ of right continuous functions J with $J(\frac{1}{2}) = 0$ is no restriction because $\bar{J}$ defined by $\bar{J}(x) = J(x+) = \lim_{y\downarrow x} J(y)$ is a well-defined right continuous function and if $\bar{J}$ belongs to $\mathcal{H}_\delta$, $\mathcal{S}_n(J)$ and $\mathcal{S}_n(\bar{J})$ are asymptotically equivalent (see [4]).

We consider the array $G_{ni}(1 \leqq i \leqq n,\ n \geqq 1)$ of d.f.'s on [0, 1] defined by

$$G_{ni} = F_{ni} \circ F_n^{-1}. \tag{2.1}$$

Denote by $\widehat{G}_n$ the empirical process in [0, 1] derived from $\widehat{H}_n$ and defined by

$$\widehat{G}_n(t) = n^{-1} \sum_{i=1}^n c_{ni} \mathbf{1}_{[F_n(X_{ni}) \leqq t]}, \qquad t \in [0,\ 1], \tag{2.2}$$

and $\hat{I}_n$ the empirical process on [0, 1] derived from $\widehat{F}_n$ and defined by

$$\hat{I}_n(t) = n^{-1} \sum_{i=1}^n \mathbf{1}_{[F_n(X_{ni}) \leqq t]}, \qquad t \in [0,\ 1]. \tag{2.3}$$

We also denote $G_n = \mathbf{E}(\widehat{G}_n)$ and $I_n = \mathbf{E}(\hat{I}_n)$. The linear rank statistic defined in (1.2) can then be written as

$$\mathcal{S}_n(J) = n^{1/2} \int_0^1 J\Big(\frac{n}{n+1}\, \hat{I}_n(t)\Big)\, d\widehat{G}_n(t) - n^{1/2} \int_0^1 J\big(I_n(t)\big)\, dG_n(t). \tag{2.4}$$

The connection between the dependence structure of the processes and the class of functions for which asymptotic normality holds is expressed by the following condition:

$$\begin{cases} n\,\mathbf{E}\big(\widehat{G}_n(t) - G_n(t)\big)^2 \leqq C\Lambda_n^2\big(t(1-t)\big)^{1-2\eta}, \\ n\,\mathbf{E}\big(\hat{I}_n(t) - I_n(t)\big)^2 \leqq C\big(t(1-t)\big)^{1-2\eta}, \end{cases} \tag{2.5}$$

for all $t \in (0,\ 1)$ and $n \geqq 1$ where $\eta = \delta(4+2\delta)^{-1}$, $\Lambda_n = \sup_{1 \leqq i \leqq n} |g(i/n)|$ for g defined in (1.1) and C is some positive constant.

PROPOSITION 2.1. *Let $K > 0$ and $\delta \geqq 0$ be given. Then there exists a constant C_1 such that the following holds: if $\{X_{ni}\}$ is an array of r.v.'s satisfying* (2.5) *and if $\{c_{ni}\}$ are regression constants defined by* (1.1), *we have*

$$n\,\mathbf{E}\bigg\{ \int_{-\infty}^{+\infty} J\Big(\frac{n}{n+1}\, \widehat{F}_n(x)\Big)\, d\widehat{H}_n(x)$$

$$-\int_{-\infty}^{+\infty} J\Big(\frac{n}{n+1} F_n(x)\Big)\, d\widehat{H}_n(x)\Big\}^2 \leqq C_1\Lambda_n^2\|J\|_\delta^2, \tag{2.6}$$

$$n\,\mathbf{E}\Big\{\int_{-\infty}^{+\infty} J\Big(\frac{n}{n+1} F_n(x)\Big)\, d(\widehat{H}_n - H_n)(x)\Big\}^2 \leqq C_1\Lambda_n^2\|J\|_\delta^2, \tag{2.7}$$

$$n\Big\{\int_{-\infty}^{+\infty} \Big(J(F_n(x)) - J\Big(\frac{n}{n+1} F_n(x)\Big)\Big)\, dH_n(x)\Big\}^2 \leqq C_1\Lambda_n^2\|J\|_\delta^2, \tag{2.8}$$

$$n\,\mathbf{E}\,(\mathcal{S}_n(J))^2 \leqq C_1\Lambda_n^2\|J\|_\delta^2. \tag{2.9}$$

Proof. We only prove (2.6) because our method of proof is similar to that of Proposition 2 in [4]. It suffices to consider an increasing function $J \in \mathcal{H}_\delta$. Define

$$\varphi(x,\ t) = \begin{cases} 1 & \text{if } \dfrac{n}{n+1} F_n(x) \leqq t < \dfrac{n}{n+1} \widehat{F}_n(x), \\ -1 & \text{if } \dfrac{n}{n+1} \widehat{F}_n(x) \leqq t < \dfrac{n}{n+1} F_n(x), \\ 0 & \text{otherwise} \end{cases}$$

and denote by $\widehat{F}_n^{-1}(t) = \inf\{x \in \mathbf{R}\colon \widehat{F}_n(x) \geq t\}$ the left continuous inverse of $\widehat{F}_n$. Since for $t \leqq n/(n+1)$,

$$\varphi(x,\ t) = \begin{cases} 1 & \text{if } \widehat{F}_n^{-1}\Big(\dfrac{n+1}{n} t\Big) \leqq x < F_n^{-1}\Big(\dfrac{n+1}{n} t\Big), \\ -1 & \text{if } F_n^{-1}\Big(\dfrac{n+1}{n} t\Big) \leqq x < \widehat{F}_n^{-1}\Big(\dfrac{n+1}{n} t\Big), \\ 0 & \text{otherwise} \end{cases}$$

it follows that for fixed $t \leqq n/(n+1)$,

$$\begin{aligned} \Big|\int_{-\infty}^{\infty} \varphi(x,\ t)\, d\widehat{H}_n(x)\Big| &\leqq \Lambda_n \int_{-\infty}^{\infty} |\varphi(x,\ t)|\, d\widehat{F}_n(x) \\ &= \Lambda_n \Big|\Big\langle \frac{n+1}{n} t\Big\rangle - \widehat{F}_n\Big(F_n^{-1}\Big(\frac{n+1}{n} t\Big) -\Big)\Big| \\ &\leqq \Lambda_n \bigg(\Big|\widehat{F}_n\Big(F_n^{-1}\Big(\frac{n+1}{n} t\Big) -\Big) - \frac{n+1}{n} t\Big| + 2\inf\big\{1/n,\ t\big\}\bigg) \end{aligned}$$

where $\langle t\rangle = (k-1)/n$ if $(k-1)/n < t \leqq k/n$.

From the assumption (2.5) on the sequence $\{X_{ni}\}$, it follows for $t \leqq n/(n+1)$ that

$$\begin{aligned} \Big(\int_{-\infty}^{+\infty} \varphi(x,\ t)\, d\widehat{H}_n(x)\Big)^2 &\leqq 8\Lambda_n^2\Big(\frac{1}{n}(1 \wedge nt)\Big)^2 + 2\,\mathbf{E}\,\Big(\hat{I}_n\Big(\frac{n+1}{n} t\Big) - I_n\Big(\frac{n+1}{n} t\Big)\Big)^2 \\ &\leqq 8\Lambda_n^2\Big(\frac{1}{n}(1 \wedge nt)\Big)^2 + 2\,\frac{C}{n}\Lambda_n^2\big(t(1-t)\big)^{1-2\eta}. \end{aligned}$$

Finally, interchanging the order of integration, and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&n\,\mathbf{E}\left(\int_{-\infty}^{+\infty} J\left(\frac{n}{n+1}\widehat{F}_n(x)\right)d\widehat{H}_n(x)-\int_{-\infty}^{+\infty} J\left(\frac{n}{n+1}F_n(x)\right)d\widehat{H}_n(x)\right)^2\\
&=n\,\mathbf{E}\left(\int_0^{n/(n+1)}\int_{-\infty}^{+\infty}\varphi(x,\ t)\,d\widehat{H}_n(x)dJ(t)\right)^2\\
&\leqq n\left(\int_0^{n/(n+1)}\left(\mathbf{E}\left(\int_{-\infty}^{+\infty}\varphi(x,\ t)\,d\widehat{H}(x)\right)^2\right)^{1/2}dJ(t)\right)^2\\
&\leqq nC_1\Lambda_n^2\left(\int_0^{n/(n+1)}\frac{1}{n}(1\wedge nt)\,dJ(t)\right.\\
&\qquad\left.+\int_0^{n/(n+1)}n^{-1/2}\big(t(1-t)\big)^{1/2-\eta}dJ(t)\right)^2\leqq C_1\Lambda_n^2\|J\|_\delta^2
\end{aligned}$$

where C_1 is some positive constant, since

$$n^{1/2}\int_0^{n/(n+1)}(1/n)\,(1\wedge nt)dJ(t)\leqq C_1\int_0^1\big(t(1-t)\big)^{1/2}dJ(t).$$

The inequality (2.6) is proved.

PROPOSITION 2.2. *Let $\{X_{ni}\}$ satisfy condition (2.5) for some $\delta>0$, and let the regression constants c_{ni} satisfy (1.1) and $\sup_{n\in\mathbf{N}}\Lambda_n<+\infty$. Assume that $K\subset\mathcal{H}_\delta$ is a subset possessing the following property: for every $J\in K$, there exists a normal distribution $\mathcal{N}(0,\ \sigma^2)$ where $0<\sigma<+\infty$ such that $\mathcal{S}_n(J)$ converges in law to $\mathcal{N}(0,\ \sigma^2)$, then the $\|\ \|_\delta$-norm closure of K has the same property.*

Proof. Let $J_1\in\overline{K}$ where $\overline{K}$ is the closure of K, and $J\in K$. By Proposition 2.1 and the fact that $\mathcal{S}_n(J_1)=\mathcal{S}_n(J)+\mathcal{S}_n(J_1-J)$, the distributions of $\mathcal{S}_n(J_1)$ and $\mathcal{S}_n(J)$ are closed, uniformly in n, in the weak topology for sufficiently small $\|J-J_1\|_\delta$. This proves the proposition.

For two probability measures $\mathbf{P}$ and $\mathbf{Q}$ on $\mathbf{R}$, denote by $D_2(\mathbf{P}\,,\ \mathbf{Q})=\inf(\mathbf{E}\,(X-Y)^2)^{1/2}$ where the infinitum extends over all r.v.'s X and Y defined on the same probability space and having distributions $\mathbf{P}$ and $\mathbf{Q}$, respectively.

Let $\mathcal{L}(Z)$ denote the distribution of the r.v. Z.

PROPOSITION 2.3. *Let $\{X_{ni}\}$ satisfy condition (2.5) for some $\delta>0$ and let c_{ni} satisfy (1.1) and $\sup_{n\in\mathbf{N}^*}\Lambda_n<+\infty$. Assume that there exists an operator σ: $\mathcal{H}_\delta\to\mathbf{R}$ which is uniformly bounded and satisfies the Lipschitz condition for the $\|\ \|_\delta$ norm.*

If for every $J\in K\subset\mathcal{H}_\delta$, $\mathcal{S}_n(J)$ converges in law to a normal distribution $\mathcal{N}(0,\ \sigma^2(J))$, then the $\|\|_\delta$-norm closure of K has the same property.

Proof. Let $J_1\in\overline{K}$ and $J\in K$. Then, we have

$$\begin{aligned}
&D_2\Big(\mathcal{L}\big(\mathcal{S}_n(J_1)\big),\ \mathcal{N}\big(0,\ \sigma^2(J_1)\big)\Big)\leqq D_2\Big(\mathcal{L}\big(\mathcal{S}_n(J_1)\big),\ \mathcal{L}\big(\mathcal{S}_n(J)\big)\Big)\\
&\qquad+D_2\Big(\mathcal{L}\big(\mathcal{S}_n(J)\big),\ \mathcal{N}\big(0,\ \sigma^2(J)\big)\Big)+D_2\Big(\mathcal{N}\big(0,\ \sigma^2(J)\big),\ \mathcal{N}\big(0,\ \sigma^2(J_1)\big)\Big)\\
&\leqq D_2\Big(\mathcal{L}\big(\mathcal{S}_n(J)\big),\ \mathcal{N}\big(0,\ \sigma^2(J)\big)\Big)+C'\|J-J_1\|_\delta
\end{aligned}$$

for some $C'>0$ using Proposition 2.1 and the Lipschitz condition on σ.

Since the convergence in law from $\mathcal{S}_n(J)$ to $\mathcal{N}(0,\ \sigma^2(J))$ implies (see [4])

$$\lim_{n\to\infty}D_2\Big(\mathcal{L}\big(\mathcal{S}_n(J)\big),\ \mathcal{N}\big(0,\ \sigma^2(J_1)\big)\Big)=0,$$

the theorem follows.

3. Convergence of the linear rank statistic. Recall that the sequence $\{X_{ni}\}$ is φ-mixing if

$$\sup_{n\geqq 1}\sup_{1\leqq j\leqq n-m}\Big\{\sup\Big\{\Big|\mathbf{P}(B|A)-\mathbf{P}(B)\Big|;\ A\in\sigma\big(X_{ni},\ 1\leqq i\leqq j\big),\ B\in\sigma\big(X_{ni},\ i\geqq j+m\big)\Big\}\Big\}=\varphi(m)\downarrow 0$$

for positive integers j and m.

Here $\sigma(X_{n1},\ldots,X_{nj})$ and $\sigma(X_{n,j+m},\ldots,X_{nn})$ are the σ-fields generated by $(X_{n1},\ldots,X_{nj})$ and $(X_{n,j+m},\ldots,X_{nn})$, respectively.

Also recall that $\{X_{ni}\}$ satisfies the strong mixing condition if

$$\sup_{n\geqq 1}\sup_{1\leqq j\leqq n-m}\Big\{\sup\Big\{\Big|\mathbf{P}(A\cap B)-\mathbf{P}(A)\,\mathbf{P}(B)\Big|;\ A\in\sigma\big(X_{ni},\ 1\leqq i\leqq j\big),\ B\in\sigma\big(X_{ni},\ i\geqq j+m\big)\Big\}\Big\}=\alpha(m)\downarrow 0.$$

Since $\alpha(m)\leqq\varphi(m)$, it follows that if $\{X_{ni}\}$ is φ-mixing, then it is also strong mixing.

We will study the asymptotic behavior of $\mathcal{S}_n(J)$ when the r.v.'s $\{X_{ni}\}$ are φ-mixing with rates

$$\sum_{m=1}^{+\infty} m\big(\varphi(m)\big)^{(2+3\delta)/(4+2\delta)}<+\infty\qquad\text{for some}\quad 0\leqq\delta<2 \tag{3.1}$$

or strong mixing with rates

$$\sum_{m=1}^{+\infty} m^2\alpha(m)^{\delta/(2+\delta)}<+\infty\qquad\text{for some}\quad \delta>0. \tag{3.2}$$

Let $F_{n,i,j}$ be the d.f. of (X_{ni},X_{nj}), $1\leqq i\leqq j\leqq n$, $n\geqq 1$. For any sequence of d.f.'s $\{G_l^*,\ l\geqq 1\}$ on $[0,1]^2$ with uniform marginals, we denote

$$\sigma_J^2(\{G_l^*\})=\lim_{n\to\infty}\left\{\int_0^1 f^2(u)\,du+2\sum_{l=1}^{n}\int_0^1\int_0^1 f(u)f(v)\,dG_l^*(u,v)\right\} \tag{3.3}$$

if the limit exists, where

$$f(u)=\int_0^1\big(\mathbf{1}_{[u\leqq v]}-v\big)\,dJ(v)+J(u)-\int_0^1 J(v)\,dv \tag{3.4}$$

and $J\in\mathcal{H}_\delta$ for some $\delta>0$.

THEOREM 3.1. *Suppose the sequence $\{X_{ni}\}$ is φ-mixing with rate* (3.1) *or strong mixing with rate* (3.2), *the function g which defines the regression constants in* (1.1) *belongs to $\mathbf{C}_{1,b}^*$ the space of functions which admit a derivative of bounded variation, and suppose that for any $l>1$, there exists a continuous d.f. G_l^* on $[0,1]^2$ with uniform marginals such that*

$$\lim_{n\to\infty}\max_{1\leqq i<j\leqq n}\Big|F_{n,i,j}\big(F_n^{-1}(u),\ F_n^{-1}(v)\big)-G_{j-i}^*(u,v)\Big|=0 \tag{3.5}$$

for all $(u,\ v)\ [0,\ 1]^2$.

Then, for every $J \in \mathcal{G}_\delta$ *with* $2 > \delta \geqq 0$ *if we have* (3.1), *and* $\delta > 0$ *if we have* (3.2)

$$\lim_{n\to\infty} D_2\Big(\mathcal{L}(\mathcal{S}_n(J)),\ \mathcal{N}(0,\ \tilde{\sigma}_J^2(\{G_l^*\}))\Big) = 0 \tag{3.6}$$

where

$$\tilde{\sigma}_J^2(\{G_l^*\}) = \sigma_J^2(\{G_l^*\}) \times \left(\int_0^1\int_0^1 (u\wedge v)\, g'(u)\, g'(v)\, dudv - 2\int_0^1 ug'(u)\, du + g^2(1)\right) \tag{3.7}$$

and $\tilde{\sigma}_J^2(\{G_l^*\}) < \infty$.

Remark 3.1. Let the sequence of distribution functions $\{F_{n,i,j}\}$ satisfy the following conditions:

(i) There exists a sequence of d.f.'s F_l^* and $\mathbf{R}^2$ such that

$$\lim_{n\to\infty}\max_{1\leqq i<j\leqq n}\Big|F_{n,i,j}(x_1,\ x_2) - F_{j-i}^*(x_1,\ x_2)\Big| = 0 \qquad \text{for all} \quad (x_1,\ x_2)\in\mathbf{R}^2,$$

(ii) $F_{ni} = F_n$ for all $1 \leqq i \leqq n,\ n \geqq 1$,

then the condition (3.5) is satisfied whenever the sequence $\{X_{ni}\}$ is strong mixing.

Proof. To prove Theorem 3.1, we first need a few lemmas.

For any n $(n \geqq 1)$, and i $(1 \leqq i \leqq n)$ and any $J \in \mathbf{C}_{2,b}$, let

$$\begin{aligned} A_{ni}(J) = &\int_0^1 \Big(\mathbf{1}_{[F_n(X_{ni})\leqq t]} - G_{ni}(t)\Big) J'(t)\, dG_{ni}(t) \\ &+ J\big(F_n(X_{ni})\big) - \int_0^1 J(t)\, dG_{ni}(t). \end{aligned} \tag{3.8}$$

It is obvious that $\mathbf{E}\,(A_{ni}) = 0$.

Now consider for any $J \in \mathbf{C}_{2,b}$, the process $L_n(J)(s)$ defined on $\mathbf{C}_1$ the space of continuous functions on $[0,\ 1]$, by

$$L_n(J)(s) = n^{-1/2}\left(\sum_{i=1}^{[ns]} A_{ni} + \big(ns - [ns]\big)\, A_{n,[ns]+1}\right), \tag{3.9}$$

where $[ns]$ denotes the integer part of the real number ns.

LEMMA 3.1. *Suppose that* $\{X_{ni}\}$ *satisfies the conditions of Theorem* 3.1 *and* J *belongs to* $\mathbf{C}_{2,b}$, *then the process* $L_n(J)(s)$ *converges weakly in uniform topology to a Gaussian process* $L_0(J)(s)$ *with trajectories a.s. in* $\mathbf{C}_1$ *with mean* 0 *and variance* $s\sigma_J^2(\{G_l^*\})$ *where* $\sigma_J^2(\{G_l^*\})$ *is defined in* (3.3), *and* $\sigma_J^2(\{G_l^*\}) < \infty$.

Proof. The process $L_n(J)$ defines a probability measure $\mathbf{P}_n$ on $\mathbf{C}_1$. From Theorem 8.1 of [2] we have to prove that (i) the finite-dimensional distribution of $\mathbf{P}_n$ converges in law to a normal distribution and (ii) $\mathbf{P}_n$ is tight.

First we prove (i), which is equivalent to proving that $\sum_{l=1}^p \lambda_l L_n(J)(s_l)$ converges in law to a normal distribution for any $p \in \mathbf{N}^*$, any $l \in [0,\ 1]$ and any $\lambda_l \in \mathbf{R}$ $(1 \leqq l \leqq p)$. Without loss of generality, we can take $p = 2$ and suppose that $s_1 < s_2$.

We have

$$\sum_{l=1}^{2} \lambda_l L_n(J)(s_l) = n^{-1/2}\Bigg[\sum_{i=1}^{[ns_1]} (\lambda_1+\lambda_2)\, A_{ni}(J) + \sum_{i=[ns_1]+1}^{[ns_2]} \lambda_2 A_{ni}(J)$$

$$+ \lambda_1\big(ns_1 - [ns_1]\big)\, A_{n,[ns_1]+1}(J)$$

$$+ \lambda_2\big(ns_2 - [ns_2]\big)\, A_{n,[ns_2]+1}(J)\Bigg]. \tag{3.10}$$

We define the sequence of r.v.'s $\{B_{ni}(J)\}$ by

$$B_{ni}(J) = \begin{cases} (\lambda_2+\lambda_2)\, A_{ni}(J) & \text{if } i \leqq [ns_1], \\ \lambda_2 A_{ni}(J) & \text{if } [ns_1] < i \leqq [ns_2], \\ 0 & \text{if } i > [ns_2]. \end{cases} \tag{3.11}$$

As J and J' are bounded, we deduce

$$\sum_{l=1}^{2} \lambda_l L_n(J)(s_l) = n^{-1/2} \sum_{i=1}^{n} B_{ni}(J) + O(n^{-1/2}). \tag{3.12}$$

From [11, Corollary 1] we have to verify that

$$\mathbf{E}\,\frac{1}{n}\left(\sum_{i=1}^{n} B_{ni}(J)\right)^2 \longrightarrow \Big\{(\lambda_1+\lambda_2)^2 s_1 + \lambda_2^2(s_2-s_1)\Big\}\,\sigma_J^2(\{G_l^*\})$$

$$\text{as} \quad n\to\infty. \tag{3.13}$$

We have

$$\mathbf{E}\,\frac{1}{n}\left(\sum_{i=1}^{n} B_{ni}(J)\right)^2 = \frac{1}{n}\Bigg[(\lambda_1+\lambda_2)^2 \sum_{i=1}^{[ns_1]}\sum_{j=1}^{[ns_1]} \mathbf{E}\,\big(A_{ni}(J)\,A_{nj}(J)\big)$$

$$+ (\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[ns_1]}\sum_{j=[ns_1]+1}^{[ns_2]} \mathbf{E}\,\big(A_{ni}(J)\,A_{nj}(J)\big)$$

$$+ \lambda_2^2 \sum_{i=[ns_1]+1}^{[ns_2]}\sum_{j=[ns_1]+1}^{[ns_2]} \mathbf{E}\,\big(A_{ni}(J)\,A_{nj}(J)\big)\Bigg]. \tag{3.14}$$

Suppose the sequence $\{X_{ni}\}$ is φ-mixing, then from the boundedness of J and J' and from the well-known inequality on the moment of φ-mixing r.v.'s (see [5, Prop. 2.2]), we obtain

$$\frac{1}{n}\left|(\lambda_1+\lambda_2)\lambda_2 \sum_{i=1}^{[ns_1]}\sum_{j=[ns_1]+1}^{[ns_2]} \mathbf{E}\,\big(A_{ni}(J)\,A_{nj}(J)\big)\right|$$

$$\leqq \frac{M}{n} \sum_{i=1}^{[ns_1]}\sum_{j=[ns_1]+1}^{[ns_2]} \varphi^{1/p}(j-i) \tag{3.15}$$

where M is some positive constant and $p = (4+2\delta)/(2+3\delta)$. From (3.1), the last expression goes to zero as $n \to \infty$.

If the sequence $\{X_{ni}\}$ is strong mixing, the left-hand side of (3.15) is majorized by

$$\frac{M'}{n} \sum_{i=1}^{[ns_1]} \sum_{j=[ns_1]+1}^{[ns_2]} \alpha^{\delta/(2+\delta)}(j-i),$$

where $M' > 0$ is some constant and from (3.2), this converges to 0 as $n \to \infty$.

It remains to prove that

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} \mathbf{E}\left(A_{ni}(J)\, A_{nj}(J)\right) \longrightarrow s_1 \sigma_J^2(\{G_l^*\}) \\ &\qquad\qquad \text{as} \quad n \to \infty, \\ &\frac{1}{n} \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} \mathbf{E}\left(A_{ni}(J)\, A_{nj}(J)\right) \longrightarrow (s_2 - s_1)\sigma_J^2(\{G_l^*\}) \\ &\qquad\qquad \text{as} \quad n \to \infty. \end{aligned} \tag{3.16}$$

We first prove the convergence of (3.16) when J and J' are replaced by indicator functions. Suppose

$$J(t) = \mathbf{1}_{[a \leqq t \leqq b]} \qquad \text{and} \qquad J' = \mathbf{1}_{[a' \leqq t \leqq b']}.$$

Then we can write

$$\begin{aligned} A_{ni}(J) &= G_{ni}(b') - G_{ni}\Big(\big\{a' \vee F_n(X_{ni})\big\} \wedge b'\Big) \\ &\quad - \int_a^b t\, dG_{ni}(t) + \mathbf{1}_{[a \leqq F_n(X_{ni}) \leqq b]} - \big(G_{ni}(b) - G_{ni}(a)\big) = D_{ni}(X_{ni}^*) \end{aligned}$$

where $X_{ni}^* = F_n(X_{ni})$. Let $G_{n,i,j}$ be the d.f. of $(F_n(X_{ni}), F_n(X_{nj}))$. Then, we have

$$\mathbf{E}\left(A_{ni}(J) A_{nj}(J)\right) = \int_0^1 \int_0^1 D_{ni}(u)\, D_{nj}(v)\, dG_{n,i,j}(u,\ v).$$

From condition (3.5) we easily deduce that

$$\lim_{n\to\infty} \max_{1 \leqq i < j \leqq n} \left| \mathbf{E}\left(A_{ni}(J) A_{nj}(J)\right) - \int_0^1 \int_0^1 D(u) D(v)\, dG_{j-i}^*(u,\ v) \right| = 0 \tag{3.17}$$

where

$$D(u) = \mathbf{1}_{[a \leqq u \leqq b]} - (b-a) + b' - \left\{(a' \vee u) \wedge b'\right\} - \tfrac{1}{2}(b'-a')^2.$$

We obtain the same result if J and J' are replaced by step functions.

As J and J' are continuous and bounded, we can uniformly approach them by step functions and we deduce that

$$\lim_{n\to\infty} \max_{1 \leqq i < j \leqq n} \left| \mathbf{E}\left(A_{ni}(J) A_{nj}(J)\right) - \int_0^1 \int_0^1 f(u) f(v)\, dG_{j-i}^*(u,\ v) \right| = 0 \tag{3.18}$$

where $f(u)$ is defined in (3.4).

Now denote $\rho(0) = \int_0^1 f^2(u)du$ and $\rho(i) = 2\int_0^1 \int_0^1 f(u)f(v)\,dG_i^*(u,v)$, $i \geqq 1$. Then,

$$\left| n^{-1} \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} \mathbf{E}\left(A_{ni}(J)\, A_{nj}(J)\right) - [ns_1]\, n^{-1} \sum_{i=0}^{+\infty} \rho(i) \right|$$

$$\leqq \left| n^{-1}[ns_1]\left([ns_1]\right)^{-1} \sum_{i=0}^{[ns_1]-1} \sum_{j=1}^{[ns_1]-i} \mathbf{E}\left(A_{nj}(J)\, A_{n,j+i}(J)\right) \right.$$

$$\left. - [ns_1]\left(n[ns_1]\right)^{-1} \sum_{i=0}^{[ns_1]} \left([ns_1]-i\right)\rho(i) \right| + [ns_1]n^{-1} \sum_{i=[ns_1]+1}^{\infty} |\rho(i)|$$

$$+ [ns_1]n^{-1} \sum_{i=0}^{[ns_1]} \sum_{k=i}^{\infty} |\rho(k)| = |A_n| + B_n + C_n.$$

From (3.18) we deduce that $|A_n| \to 0$ as $n \to \infty$ and from the well-known inequalities on the moment of mixing r.v.'s (see [5, Props. 2.2 and 2.8]) and (3.1) or (3.2) we deduce that $B_n \to 0$ and $C_n \to 0$ and $n \to \infty$.

It is also immediate that

$$\left| [ns_1]n^{-1} \sum_{i=0}^{+\infty} \rho(i) - s_1 \sum_{i=0}^{+\infty} \rho(i) \right| \longrightarrow 0 \qquad \text{as} \quad n \to \infty.$$

We conclude that $n^{-1} \sum_{i=1}^{[ns_1]} \sum_{j=1}^{[ns_1]} \mathbf{E}\left(A_{ni}(J)A_{nj}(J)\right)$ converges to $s_1(\sum_{i=0}^{+\infty} \rho(i))$ as $n \to \infty$ where $\sum_{i=0}^{+\infty} \rho(i)$ is equal to $\sigma_J^2(\{G_l^*\})$. Similarly

$$\frac{1}{n} \sum_{i=[ns_1]+1}^{[ns_2]} \sum_{j=[ns_1]+1}^{[ns_2]} \mathbf{E}\left(A_{ni}(J)\, A_{nj}(J)\right) \longrightarrow (s_2 - s_1)\sigma_J^2(\{G_l^*\})$$

as $n \to \infty$, so (3.16) is proved.

From (3.14)–(3.16), we deduce (3.13) and we conclude that $\mathbf{E}\left(\sum_{l=1}^{2} \lambda_l L_n(J)(s_l)\right)^2$ converges to $\{(\lambda_1+\lambda_2)^2 s_1 + \lambda_2^2(s_2 - s_1)\}\sigma_J^2(\{G_l^*\})$ which implies that $\sum_{l=1}^{2} \lambda_l L_n(J)(s_l)$ converges in law to the normal distribution with mean 0 and variance $\{(\lambda_1+\lambda_2)^2 s_1 + \lambda_2^2(s_2 - s_1)\}\sigma_J^2(\{G_l^*\})$ and (i) is proved.

We now prove (ii).

From [2, Thm. 8.2] we have to verify that $\forall \varepsilon > 0\ \exists \eta > 0$ $(0 < \eta < 1)$ and an integer N_0 such that $\forall n \geqq N_0$

$$\mathbf{P}\left[\sup_{|s-s'|<\eta} \left| L_n(J)(s) - L_n(J)(s') \right| \geqq \varepsilon \right] \leqq \varepsilon. \tag{3.19}$$

If ns and ns' are integers, by using Theorem 11 from [5] for $q = 4$ for the strong mixing case and Lemma 5.1 in Harel (1988) for $q = 2$ for the φ-mixing case, we obtain for $s \geqq s'$

$$\mathbf{E}\left(L_n(J)(s) - L_n(J)(s')\right)^4 \leqq \left((s-s')^2 + n^{-1}(s-s')\right) MC(\beta) \tag{3.20}$$

where

$$C(\beta) = \sum_{m=1}^{+\infty} m^{-1}\varphi^{1/4}(m) \tag{3.21}$$

if the sequence $\{X_{ni}\}$ is φ-mixing, and

$$C(\beta) = \sum_{m=1}^{+\infty} m^2 \alpha^{\delta/(2+\delta)}(m) \tag{3.22}$$

if the sequence $\{X_{ni}\}$ is strong mixing and M is some positive constant.

If $s > s'$ and ns and ns' are integers, we have $s - s' \geqq n^{-1}$ and

$$\mathbf{E}\left(L_n(J)(s) - L_n(J)(s')\right)^4 \leqq 2M(s-s')^2 C(\beta).$$

From [1, Lemma 2] we obtain that for any $\varepsilon > 0$ there exist $\eta > 0$, and an integer N_0 sufficiently large such that $\forall n \geqq N_0$,

$$\begin{aligned} &\mathbf{P}\left[\sup_{|[ns]/n-[ns']/n|<2\eta} \left|L_n(J)\big([ns]/n\big) - L_n(J)\big([ns']/n\big)\right| > \varepsilon/2\right] \\ &\quad \leqq 2MC(\beta)K\eta^2\varepsilon^{-4} \end{aligned} \tag{3.23}$$

where K is some positive constant.

From the definition of $L_n(J)(s)$ in (3.9), we obtain

$$\begin{aligned} &\sup_{|s-s'|<\eta} \left|L_n(J)(s) - L_n(J)(s')\right| \\ &\quad \leqq 2 \max_{|[ns]/n-[ns']/n|<2\eta} \left|L_n(J)\big([ns]/n\big) - L_n(J)\big([ns']/n\big)\right|. \end{aligned} \tag{3.24}$$

By using (3.23) and (3.24), we deduce

$$\mathbf{P}\left[\sup_{|s-s'|<\eta} \left|L_n(J)(s) - L_n(J)(s')\right| \geqq \varepsilon\right] \leqq MC(\beta)K'\eta^2\varepsilon^{-4} \tag{3.25}$$

where K' is some positive constant and (3.19) is proved. The fact that $\sigma_J^2(\{G_l^*\}) < \infty$ is a simple consequence of $J \in \mathbf{C}_{2,b}$ and (3.1) or (3.2).

Now we consider for any $J \in \mathbf{C}_{2,b}$ the r.v. $V_n(J)$ defined by

$$V_n(J) = n^{-1/2} \sum_{i=1}^{n} c_{ni} A_{ni}. \tag{3.26}$$

LEMMA 3.2. *Suppose that $\{X_{ni}\}$ satisfies the conditions of Theorem 3.1, J belongs to $\mathbf{C}_{2,b}$ and g admits a derivative g'. Then $V_n(J)$ converges in law to the normal distribution with mean 0 and variance $\tilde{\sigma}_J^2(\{G_l^*\})$ where $\tilde{\sigma}_J^2(\{G_l^*\})$ is defined in (3.7) and $\tilde{\sigma}_J^2(\{G_l^*\}) < \infty$.*

Proof. For any n define a measure λ_n on $[0, 1]$ by setting

$$\lambda_n\big(\{i/n\}\big) = c_{ni} - c_{n,i+1}, \qquad 1 \leqq i \leqq n-1, \quad \text{and} \quad \lambda_n\big(\{1\}\big) = c_{nn}.$$

By definition, we have

$$V_n(J) = \int_0^1 L_n(J)(u)\lambda_n(du).$$

We now prove that

$$\int_0^1 L_n(J)(u)\lambda_n(du) \qquad \text{converges in law to}$$

$$-\int_0^1 L_0(J)(u)g'(u)du + L_0(J)(1)g(1) \qquad \text{as} \quad n\to\infty. \tag{3.27}$$

Let h_n: $\mathbf{C}_1 \to \mathbf{R}$, $n \geqq 1$, be defined as $h_n(f) = \int_0^1 f(u)\lambda_n(du)$ and h_0: $\mathbf{C}_1 \to \mathbf{R}$ be defined as $h_0(f) = -\int_0^1 fg'(u)du + f(1)g(1)$. Let $\{f_n,\ n \geqq 1\}$ be a sequence of functions in $\mathbf{C}_1$ and suppose that $f_n \to f_0$ in uniform topology as $n\to\infty$ where $f_0 \in \mathbf{C}_1$. We show that $h_n(f_n) \to h_0(f_0)$ as $n\to\infty$.

$$\begin{aligned}
&\left|\int_0^1 f_n(u)\lambda_n(du) - \left(-\int_0^1 f_0(u)g'(u) + f_0(1)g(1)\right)\right| \\
&\quad\leqq \left|\int_0^1 \big(f_n(u)-f_0(u)\big)\lambda_n(du)\right| \\
&\qquad + \left|\int_0^1 f_0(u)\lambda_n(du) + \int_0^1 f_0(u)g'(u) - f_0(1)g(1)\right| \\
&\quad\leqq \sup_{u\in[0,1]} \big|f_n(u)-f_0(u)\big| \left|\int_0^1 \lambda_n(du)\right| \\
&\qquad + \left|\sum_{i=1}^{n-1} f_0(i/n)\Big(g(i/n) - g\big((i+1)/n\big)\Big) + f_0(1)g(1)\right. \\
&\qquad\quad \left. + \int_0^1 f_0(u)g'(u)\,du - f_0(1)g(1)\right| \longrightarrow 0 \qquad \text{as} \quad n\to\infty
\end{aligned}$$

from the hypothesis $f_n \to f_0$ in uniform topology, g' is an integrable function and $\sup_{n\in\mathbf{N}^*}\Lambda_n < \infty$.

Consequently, $h_n(f_n) \to h_0(f_0)$ as $n\to\infty$ and by [2, Thm. 5.5] (3.27) follows.

It remains to show that

$$\mathbf{E}\left(-\int_0^1 L_0(J)(u)g'(u)du + L_0(J)(1)g(1)\right)^2 = \widetilde{\sigma}_J\big(\{G_l^*\}\big) < \infty.$$

We have

$$\begin{aligned}
&\mathbf{E}\left(-\int_0^1 L_0(J)(u)g'(u)\,du + L_0(J)(1)g(1)\right)^2 \\
&\quad= \int_0^1\int_0^1 \mathbf{E}\left[L_0(J)(u)L_0(J)(v)\right] g'(u)g'(v)\,du\,dv \\
&\qquad - 2\int_0^1 \mathbf{E}\left[L_0(J)(1)L_0(J)(u)\right] g'(u)\,du + \mathbf{E}\left[L_0(J)(1)L_0(J)(1)\right] g^2(1).
\end{aligned}$$

As $\mathbf{E}\left[L_0(J)(u)L_0(J)(v)\right] = (u\wedge v)\sigma_J^2(\{G_l^*\}) < \infty$, the property follows and Lemma 3.2 is proved.

Proof of the theorem. We first prove that the theorem is true for $J \in \mathbf{C}_{2,b}$. We have the following decomposition:

$$\mathcal{S}_n(J) = V_n(J) + A_n(J) + B_n(J) + C_n(J),$$

where

$$A_n(J) = n^{1/2} \int_0^1 J'\big(I_n(t)\big)\big(\hat{I}_n(t) - I_n(t)\big)\, d\big(\widehat{G}_n - G_n\big)(t),$$
$$B_n(J) = -(n+1)^{-1} n^{1/2} \int_0^1 J'\big(I_n(t)\big)\hat{I}_n(t)\, d\widehat{G}_n(t),$$
$$C_n(J) = 2^{-1} n^{1/2} \int_0^1 J''\Big(\theta_n\big(I_n(t)\big)\Big)\Big(n(n+1)^{-1}\big(\hat{I}_n(t) - I_n(t)\big)\Big)^2 d\widehat{G}_n(t),$$

where $\theta_n(I_n(t)) \in [I_n(t) \wedge \hat{I}_n(t),\, I_n(t) \vee \hat{I}_n(t)]$.

Suppose $J \in \mathbf{C}_{2,b}$. Then the weak convergence of $V_n(J)$ is established in Lemma 3.2. The random variables $A(J)$, $B(J)$, $C(J)$ converge to zero in probability and in $\mathbf{L}_2$, since

$$\mathbf{E}\left(A_n^2(J)\right) \leqq K n^{-1} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \sup_{n\in\mathbf{N}^*} \Lambda_n^2 C(\beta), \tag{3.28}$$

$$\left|B_n(J)\right| \leqq K n^{-1/2} \sup_{t\in[0,1]} \left|J'(t)\right| \sup_{n\in\mathbf{N}^*} \Lambda_n, \tag{3.29}$$

$$\mathbf{E}\left(C_n^2(J)\right) \leqq K n^{-1} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \sup_{n\in\mathbf{N}^*} \Lambda_n^2 C(\beta) \tag{3.30}$$

where $C(\beta) = \sum_{m=1}^{+\infty} m(\varphi(m))^{(2+3\delta)/(4+2\delta)}$ if we have (3.1), and $C(\beta) = m^2 \times \sum_{m=1}^{+\infty} (\alpha(m))^{\delta/(2+\delta)}$ if we have (3.2), and K is some positive constant.

We only prove the inequality (3.28) for the φ-mixing case, because the method is similar to the proof of the three inequalities in [4, p. 66]. For any $(i, j, l, q) \in \mathbf{N}^4$, we put

$$\beta(i, j, l, q) = \left(\int_0^1 J'\big(I_n(t)\big)c_{ni}\big(\mathbf{1}_{[X_{ni}\leqq t]} - F_{ni}(t)\big)\, d\big(\mathbf{1}_{[X_{nj}\leqq t]} - F_{nj}(t)\big)\right) \times \left(\int_0^1 J'\big(I_n(u)\big)c_{nl}\big(\mathbf{1}_{[X_{nl}\leqq u]} - F_{nl}(u)\big)\, d\big(\mathbf{1}_{[X_{nq}\leqq u]} - F_{nq}(u)\big)\right).$$

Suppose $i \leqq j \leqq l \leqq q$ and let $p = (2+3\delta)/(4+2\delta)$, then from the condition of φ-mixing, we have the following three inequalities:

$$\beta(i, j, l, q) \leqq \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2 \varphi(j-i),$$
$$\beta(i, j, l, q) \leqq \Big(2 \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2\Big)\Big(\varphi(l-j) + 4\varphi^{1/p}(j-i)\varphi^{1/p}(q-l)\Big),$$
$$\beta(i, j, l, q) \leqq \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2 \varphi(q-l).$$

If i, j, l, q are differently ordered, we obtain similar inequalities.

From this, we deduce

$$\mathbf{E}\left(A_n^2(J)\right) \leqq 4!\, n^{-3} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2 \sum_{i,j,l,q} \beta(i, j, l, q).$$

Put $j' = j - i$, $l' = l - j$ and $q' = q - l$. We have

$$\begin{aligned}\mathbf{E}\left(A_n^2(J)\right) &\leqq 4!\, n^{-3} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2 \\ &\times \sum_{i=1}^{n} \Bigg(\sum_{\substack{0\leqq l'\leqq j' \\ 0\leqq q'\leqq j'}} 2\varphi(j') + \sum_{\substack{0\leqq j'\leqq q' \\ 0\leqq l'\leqq q'}} 2\varphi(q') + \sum_{\substack{0\leqq j'\leqq l' \\ 0\leqq q'\leqq l'}} 2\varphi(l') + 4\varphi^{1/p}(j')\varphi^{1/p}(q') \Bigg)\end{aligned}$$

and after some computations, we obtain

$$\begin{aligned}\mathbf{E}\left(A_n^2(J)\right) &\leqq 288 n^{-3} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \Lambda_n^2 \Bigg(\sum_{m\geqq 1} \varphi^{1/p}(m) \Bigg) \Bigg(\sum_{m\geqq 1} m\varphi^{1/p}(m)/m \Bigg) (n + n^2) \\ &\leqq K n^{-1} \sup_{t\in[0,1]} \left|J'(t)\right|^2 \sup_{n\in\mathbf{N}^*} \Lambda_n^2 C(\beta)\end{aligned}$$

where K is some positive constant and $C(\beta) = \sum_{m\geqq 1} m(\varphi(m))^{(2+3\delta)/(4+2\delta)}$. The inequality (3.28) is proved for the φ-mixing case. Hence the theorem is true for $J \in \mathbf{C}_{2,b}$.

Following Proposition 2.3 it remains to prove that the operator σ: $\mathcal{H}_\delta \to \mathbf{R}$ defined by $\sigma(J) = \widetilde{\sigma}_J(\{G_l^*\})$ satisfies the Lipschitz condition for the $\|\,\|_\delta$ norm and the condition (2.5) is satisfied. The first property follows easily from the definition of $\widetilde{\sigma}_J(\{G_l^*\})$ in (3.7) and the definition of $\sigma_J(\{G_l^*\})$ in (3.3) and (3.4).

We now prove (2.5) if we have (3.1). For $p = (4+2\delta)/(2+3\delta)$ and $q = (1-p^{-1})^{-1}$ we have

$$\begin{aligned}n\,\mathbf{E}\left(\widehat{G}_n(t) - G_n(t)\right)^2 &= n\,\mathbf{E}\left[n^{-1} \sum_{i=1}^{n} c_{ni}\left(\mathbf{1}_{[F_n(X_{ni})\leqq t]} - G_{ni}(t)\right)\right]^2 \\ &= n^{-1} \sum_{1\leqq i,j\leqq n} \mathbf{E}\left[c_{ni}c_{nj}\left(\mathbf{1}_{[F_n(X_{ni})\leqq t]} - G_{ni}(t)\right)\left(\mathbf{1}_{[F_n(X_{nj})\leqq t]} - G_{nj}(t)\right)\right] \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1}\sum_{j=1}^{n-i} \mathbf{E}\left|\left(\mathbf{1}_{[F_n(X_{ni})\leqq t]} - G_{ni}(t)\right)\left(\mathbf{1}_{[F_n(X_{n,j+i})\leqq t]} - G_{n,j+1}(t)\right)\right| \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \varphi^{1/p}(i) \sum_{j=1}^{n-i} \Big(G_{n,j}(t)\big(1 - G_{n,j}(t)\big)\Big)^{1/p} \\ &\quad \times \Big(G_{n,j+i}(t)\big(1 - G_{n,j+i}(t)\big)\Big)^{1/q} \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=1}^{n-1} \varphi^{1/p}(i) \Bigg(\sum_{j=1}^{n} G_{n,j}(t)\big(1 - G_{n,j}(t)\big)\Bigg)^{1/p} \\ &\quad \times \Bigg(\sum_{j=1}^{n} G_{n,j}(t)\big(1 - G_{n,j}(t)\big)\Bigg)^{1/q} \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=1}^{n-1} \varphi^{1/p}(i) \Bigg(\sum_{j=1}^{n} G_{n,j}(t)\big(1 - G_{n,j}(t)\big)\Bigg) \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=1}^{n-1} \varphi^{1/p}(i)\, n \Bigg(\frac{1}{n}\sum_{j=1}^{n} G_{nj}(t)\Bigg)\Bigg(1 - n^{-1}\sum_{j=1}^{n} G_{nj}(t)\Bigg)\end{aligned}$$

$$= 2\Lambda_n^2 \sum_{i=1}^{n-1} \varphi^{1/p}(i)t(1-t) \leqq 2\left(\sum_{i=1}^{+\infty} \varphi^{1/p}(i)\right)\Lambda_n^2 t(1-t),$$

which implies (2.5) if we have (3.1).

Finally, we prove (2.5) if we have (3.2). We have in analogy with preceding arguments

$$\begin{aligned} n\,\mathbf{E}\left(\widehat{G}(t) - G_n(t)\right)^2 &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \left(\alpha(i)\right)^{\delta/(2+\delta)} \sum_{j=1}^{n-i} \Big(G_{nj}(t)\big(1 \\ &\quad - G_{nj}(t)\big)\Big)^{1/(2+\delta)} \Big(G_{n,j+i}(t)\big(1 - G_{n,j+i}(t)\big)\Big)^{1/(2+\delta)} \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \left(\alpha(i)\right)^{\delta/(2+\delta)} \left(\sum_{j=1}^{n} G_{nj}\big(1 - G_{nj}(t)\big)\right)^{1/(2+\delta)} \\ &\quad \times \left(\sum_{j=1}^{n} G_{n,j}(t)\big(1 - G_{n,j}(t)\big)\right)^{1/(2+\delta)} \\ &= 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \left(\alpha(i)\right)^{2/(2+\delta)} \left(\sum_{j=1}^{n} G_{nj}(t)\big(1 - G_{nj}(t)\big)\right)^{2/(2+\delta)} \\ &\leqq 2n^{-1}\Lambda_n^2 \sum_{i=0}^{n-1} \left(\alpha(i)\right)^{\delta/(2+\delta)} n\left(n^{-1}\sum_{j=1}^{n} G_{nj}(t)\left(1 - n^{-1}\sum_{j=1}^{n} G_{nj}(t)\right)\right)^{2/(2+\delta)} \\ &= 2\Lambda_n^2 \sum_{i=0}^{n-1} \left(\alpha(i)\right)^{\delta/(2+\delta)} \big(t(1-t)\big)^{2/(2+\delta)} \\ &\leqq 2\Lambda_n^2 \sum_{i=0}^{+\infty} \left(\alpha(i)\right)^{2/(2+\delta)} \big(t(1-t)\big)^{1-2\eta} \qquad \text{for} \quad \eta = \delta(4+2\delta)^{-1}. \end{aligned}$$

Thus (2.5) is proved if we have (3.2).

COROLLARY 3.1. *If among conditions in Theorem* 3.1 *the function* g *is replaced by a function for which there exists a decomposition* $g = g_c + g_d$ *where* $g_c \in \mathbf{C}_{1,b}^*$ *and* g_d *is a step function with* p *jumps, say at* $a_1, \ldots, a_p$ *such that* $a_i \in (0,\, 1)$ $(1 \leqq i \leqq p)$, *then the conclusion of Theorem* 3.1 *remains true but* $\widetilde{\sigma}_J(\{G_l^*\})$ *defined in* (3.7) *is replaced by* $\widehat{\sigma}_J^2(\{G_l^*\})$ *where*

$$\begin{aligned} \widehat{\sigma}_J^2(\{G_l^*\}) = \sigma_J^2(\{G_l^*\})\bigg(&\int_0^1\int_0^1 (u \wedge v) g_c'(u) g_c'(v)\,du\,dv \\ &- 2\sum_{i=1}^{p} \big(g_d(a_i-) - g_d(a_i+)\big)\int_0^1 (u \wedge a_i) g_c'(u)\,du \\ &- 2\int_0^1 u g_c'(u)\,du + \sum_{1<i,j\leqq p} \big(a_i \wedge a_j\big)\big(g_d(a_i-) \\ &- g_d(a_i+)\big)\big(g_d(a_j-) - g_d(a_j+)\big) + 2\sum_{i=1}^{p} a_i\big(g_d(a_i-) - g_d(a_i+)\big) + g^2(1)\bigg), \end{aligned} \tag{3.31}$$

where $\sigma_J^2(\{G_l^*\})$ *is defined in* (3.3).

4. Convergence of the two-sample linear rank statistic. Let $\{Y_{n_1 i}\}$, $1 \leqq i \leqq n_1$, and $\{Z_{n_2 j}\}$, $1 \leqq j \leqq n_2$, be two independent sequences of weakly dependent random variables with continuous d.f.'s $F^{(1)}_{n_1 i}(x)$ and $F^{(2)}_{n_2 j}(x)$, respectively, $x \in \mathbf{R}$. Given $n = n_1 + n_2$ we set $X_{ni} = Y_{n_1 i}$ if $i \leqq n_1$ and $X_{ni} = Y_{n_2 i - n_1}$ if $i > n_1$. Denote by $\widehat{F}^{(1)}_{n_1}(x) = n_1^{-1} \sum_{i=1}^{n_1} c_{n_1 i} \mathbf{1}_{[X_{ni} \leqq x]}$ the empirical process based on the first sequence of r.v.'s $\{Y_{n_1 i}\}$ and weighted by the regression constants $c_{n_1 i}$. We put $F^{(1)}_{n_1} = \mathbf{E}\,(\widehat{F}^{(1)}_{n_1})$.

Then $\mathcal{S}^*_n(J)$ defined by

$$\mathcal{S}^*_n(J) = n^{1/2}\biggl(\int_{-\infty}^{+\infty} J\left(\frac{n}{n+1}\, \widehat{F}_n(x)\right) d\widehat{F}^{(1)}_{n_1}(x) - \int_{-\infty}^{+\infty} J\left(F_n(x)\right) dF^{(1)}_{n_1}(x) \biggr) \tag{4.1}$$

is the two-sample linear rank statistic. We suppose that the regression constants $c_{n_1 i}$ ($1 \leqq i \leqq n_1$) are defined by a function h on [0, 1] as

$$c_{n_1 i} = h(i/n_1), \qquad 1 \leqq i \leqq n_1, \quad n_1 \geqq 1.$$

We assume that $n_1 n^{-1} \to \lambda_0 \in (0,\, 1)$.

We have $F_n = n^{-1}(\sum_{i=1}^{n_1} F^{(1)}_{n_1 i} + \sum_{j=1}^{n_2} F^{(2)}_{n_2 j})$.

Let $F^{(1)}_{n_1, i, l}$ be the d.f. of $(Y_{n_1 i}, Y_{n_1 l})$ and $F^{(2)}_{n_2, j, k}$ be the d.f. of $(Z_{n_2 j}, Z_{n_2 k})$.

THEOREM 4.1. *Suppose the sequences $\{Y_{n_1 i}\}$ and $\{Z_{n_2 j}\}$ are φ-mixing with rate* (3.1) *or strong mixing with rate* (3.2), *the function h satisfies $h = h_c + h_d$ with $h_c \in \mathbf{C}^*_{1,b}$ and h_d is a step function and if for each $p > 1$, there exist two continuous d.f.'s $\widehat{G}^{(1)}_p$ and $\widehat{G}^{(2)}_p$ on $\mathbf{R}^2$ with marginals $F^{(1)}$ and $F^{(2)}$ such that*

$$\lim_{n\to\infty} \max_{1 \leqq i < j \leqq n} \left| F^{(l)}_{n_l, i, j}\left(F_n^{-1}(t_1),\ F_n^{-1}(t_2)\right) - \widehat{G}^{(l)}_{j-i}\left(H^{-1}(t_1),\ H^{-1}(t_2)\right) \right| = 0 \tag{4.2}$$

for all $(t_1,\, t_2) \in [0,\, 1]^2$, $l = 1,\, 2$, where

$$H = \lambda_0 F^{(1)} + (1 - \lambda_0)\, F^{(2)}. \tag{4.3}$$

Then, for every $J \in \mathcal{G}_\delta$ with $2 > \delta \geqq 0$ if we have (3.1) *and $\delta > 0$ if we have* (3.2)

$$\lim D_2\Bigl(\mathcal{L}\bigl(\mathcal{S}^*_n(J)\bigr),\ \mathcal{N}\bigl(0,\ \widetilde{\sigma}_J(\{G^{(l)}_p\})\bigr)\Bigr) = 0 \tag{4.4}$$

where

$$\widetilde{\sigma}^2_J(\{\widehat{G}^{(l)}_p\}) = \bar{\sigma}^2_J\Bigl(\{\widehat{G}^{(l)}_p\}\Bigr)\, L(h) \tag{4.5}$$

where

$$\begin{aligned}
\bar{\sigma}^2_J\Bigl(\{G^{(l)}_p\}\Bigr) = \lambda_0^{-1}\biggl\{ &\int_0^1 f_1^2(u)\, d\bigl(F^{(1)} \circ H^{-1}\bigr)(u) \\
&+ 2 \sum_{p \geqq 2} \int_0^1 \int_0^1 f_1(u) f_1(v)\, d\Bigl(\widehat{G}^{(1)}_p\bigl(H^{-1}(u),\ H^{-1}(v)\bigr)\Bigr)\biggr\} \\
+ (1 - \lambda_0)\biggl\{ &\int_0^1 f_2^2(u)\, d\bigl(F^{(2)} \circ H^{-1}\bigr)(u) \\
&+ 2 \sum_{p \geqq 2} \int_0^1 \int_0^1 f_2(u) f_2(v)\, d\Bigl(\widehat{G}^{(2)}_p\bigl(H^{-1}(u),\ H^{-1}(v)\bigr)\Bigr)\biggr\}.
\end{aligned} \tag{4.6}$$

Here

$$f_1(u) = f_1^*(u) - \int_0^1 f_1^*(v)\, d\big(F^{(1)} \circ H^{-1}(v)\big),$$
$$f_2(u) = f_2^*(u) - \int_0^1 f_2^*(v)\, d\big(F^{(2)} \circ H^{-1}(v)\big),$$

with

$$f_1^*(u) = J(u) + \lambda_0 \int_u^1 J'(v)\, d\big(F^{(1)} \circ H^{-1}(v)\big),$$
$$f_2^*(u) = \int_u^1 J'(v)\, d\big(F^{(2)} \circ H^{-1}(v)\big),$$

and

$$\begin{aligned} L(h) = {} & \int_0^1 \int_0^1 h_c'(u) h_c'(v)\, du\, dv - 2 \sum_{i=1}^p \big(h_d(a_i-) - h_d(a_i+)\big) \int_0^1 h_c(u)\, du \\ & - 2 \int_0^1 h_c'(u)\, du + \sum_{1 \leqq i, j \leqq p} \big(h_d(a_i-) - h_d(a_i+)\big)\big(h_d(a_j-) - h_d(a_j+)\big) \\ & + 2 \sum_{i=1}^p \big(h_d(a_i-) - h_d(a_i+)\big) + h_c^2(1), \end{aligned} \tag{4.7}$$

where a_i, $1 \leqq i \leqq p$, are the discontinuous points of h_d.

Remark. Let the sequences $\{F^{(1)}_{n_1,i,j}\}$ and $\{F^{(2)}_{n_2,i,j}\}$ satisfy the following conditions:

(i) there exist two sequences of d.f.'s $G_p^{(l)}$ on $\mathbf{R}^2$, $l = 1, 2$, such that

$$\lim_{n\to\infty} \max_{1 \leqq i < j \leqq n} \big|F^{(l)}_{n_l,i,j}(x_1,\ x_2) - G^{(l)}_{j-1}(x_1,\ x_2)\big| = 0$$

for all $(x_1,\ x_2) \in \mathbf{R}^2$, $l = 1, 2$,

(ii) $F^{(1)}_{n_1 i} = F^{(1)}_{n_1}$ for all $1 \leqq i \leqq n_1$,

(iii) $F^{(2)}_{n_2 j} = F^{(2)}_{n_2}$ for all $1 \leqq j \leqq n_2$.

Then condition (4.2) is satisfied when the sequences $\{Y_{n_1 i}\}$ and $\{Z_{n_2 j}\}$ are strong mixing.

Proof of Theorem 4.2. For any $n_1 \geqq 1$, for any i $(1 \leqq i \leqq n_1)$ and for any $J \in \mathbf{C}_{2,b}$, let

$$\begin{aligned} B_{n_1 i}(J) = {} & n_1 n^{-1} \int_0^1 \Big(\mathbf{1}_{[F_n(Y_{n_1 i}) \leqq t]} - G^{(1)}_{n_1 i}(t)\Big) J'(t)\, dG^{(1)}_{n_1 i}(t) \\ & + J\Big(F_n\big(Y_{n_1 i}\big)\Big) - \int_0^1 J(t)\, dG^{(1)}_{n_1 i}(t) \end{aligned} \tag{4.8}$$

where $G^{(1)}_{n_1 i} = F^{(1)}_{n_1 i} \circ F_n^{-1}$, and for any $n_2 \geqq 1$, any j $(1 \leqq j \leqq n_2)$, any $J \in \mathbf{C}_{2,b}$ and any $u \in [0,\ 1]$, let

$$\begin{aligned} C_{n_2 j}(J)(u) = {} & n_2^{-1} \sum_{l=1}^{[n_2 u]} \int_0^1 \Big(\mathbf{1}_{[F_n(Z_{n_2 j}) \leqq t]} - G^{(2)}_{n_2 j}(t)\Big) J'(t)\, dG^{(2)}_{n_2 l}(t) \\ & + n_2^{-1}\big(n_2 u - [n_2 u]\big) \int_0^1 \Big(\mathbf{1}_{[F_n(Z_{n_2 j}) \leqq t]} - G^{(2)}_{n_2 j}(t)\Big)\, dG^{(2)}_{n_2,[n_2 u]+1}(t), \end{aligned} \tag{4.9}$$

where $G^{(2)}_{n_2 j} = F^{(2)}_{n_2 j} \circ F_n^{-1}$.

Now consider for any $J \in \mathbf{C}_{2,b}$ the processes $W_{n_1}(J)(s)$ and $W^*_{n_2}(J)(s,u)$ defined, respectively, on $\mathbf{C}_1$ and $\mathbf{C}^*_1$ (= the space of continuous functions on $[0, 1]^2$) by

$$W_{n_1}(J)(s) = n_1^{1/2}\left(\sum_{i=1}^{[n_1 s]} B_{n_1 i}(J) + \big(n_1 s - [n_1 s]\big)\, B_{n_1,[n_1 s]+1}(J)\right), \tag{4.10}$$

$$W^*_{n_2}(J)(s,\ u) = n_2^{1/2}\left(\sum_{i=1}^{[n_2 s]} C_{n_2 j}(J)(u) + \big(n_2 s - [n_2 s]\big)\, C_{n_2,[n_2 s]+1}(J)(u)\right). \tag{4.11}$$

By similar techniques as in Lemma 3.1 one can prove that the process $W_{n_1}(J)(s)$ converges weakly in uniform topology to a Gaussian process $W_0(J)(s)$ with trajectories a.s. in $\mathbf{C}_1$ with mean 0 and variance

$$\begin{aligned} s\Bigg[\int_0^1\int_0^1 & f_1^2(u)\, d\big(F^{(1)} \circ H^{-1}\big)(u) \\ & + 2\sum_{p\geqq 1}\int_0^1\int_0^1 f_1(u) f_1(v)\, d\Big(G^{(1)}_p\big(H^{-1}(u),\ H^{-1}(v)\big)\Big)\Bigg] \end{aligned} \tag{4.12}$$

and $W^*_{n_2}(J)(s,u)$ converges weakly in uniform topology to a Gaussian process $W^*_0(J)(s,\ u)$ with trajectories a.s. in $\mathbf{C}^*_1$ with mean 0 and variance

$$\begin{aligned} su\Bigg[\int_0^1 & f_2^2(u)\, d\big(F^{(2)} \circ H^{-1}\big)(u) \\ & + 2\sum_{p\geqq 1}\int_0^1\int_0^1 f_2(u) f_2(v)\, d\Big(G^{(2)}_p\big(H^{-1}(u),\ H^{-1}(v)\big)\Big)\Bigg]. \end{aligned} \tag{4.13}$$

From this and following Lemma 3.2, it is easy to prove that $V^*_n(J)$ converges in law to the normal distribution with mean 0 and variance $\widetilde{\sigma}_J(\{\widehat{G}^{(l)}_p\})$ where $\widetilde{\sigma}_J(\{\widehat{G}^{(l)}_p\})$ is defined in (4.5) and $V^*_n(J)$ is a random variable defined by

$$\begin{aligned} V^*_n(J) = n^{1/2}\Bigg(&\int_{-\infty}^{+\infty} J\big(\widehat{F}_n(x)\big)\, d\big(\widehat{F}^{(1)}_n(x) - F^{(1)}_n(x)\big) \\ & + \int_{-\infty}^{+\infty} J'\big(\widehat{F}_n(x)\big)\big(\widehat{F}_n(x) - F_n(x)\big)\, dF^{(1)}_n(x)\Bigg). \end{aligned} \tag{4.14}$$

Since $S^*_n(J) = V^*_n(J) + U_n(J)$ where $\mathbf{E}\,(U_n(J))^2 = O(n^{-1/2})$ we prove Theorem 4.1 following the line of argument in the proof of Theorem 3.1.

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